Moonshine With A Twist Of Geometry

Sean Colin-Ellerin

November 14, 2016
Outline

- What is Moonshine?
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- Groups and Representations
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▶ Modular Forms
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- Modular Forms
- Monstrous Moonshine
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- Discriminant Property
What is Moonshine?

- Geometry

Module forms

- String Theory
- Groups
- Algebras

+ Geometry
Groups and Representations

- Group = Symmetries
  - Ex. $D_8 =$ symmetries of square

\[ f_1, f_2, f_3, f_4, r_1, r_2, r_3 \]
Groups and Representations

- Group = Symmetries
  - Ex. $D_8$ = symmetries of square

- Representations give action of group on objects in terms of matrices
Modular Forms

- A torus is given by two vectors (periods) in upper-half plane $\mathcal{H}$, but one parameter $\tau$

A modular form of weight $k$ is a holomorphic map $f: \mathcal{H} \to \mathbb{C}$ that is the same on equivalent tori (up to prefactor): $f(az + b, cz + d) = (cz + d)^k f(z)$ where $(a b; c d) \in SL_2(\mathbb{Z})$.

Important in string theory because tori are good candidates for string compactifications, modular forms can be expanded in Laurent series $f(z) = \sum_{n=0}^{\infty} a_n q^n = e^{2\pi iz}$, which is the partition function.
Modular Forms

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- Remains untwisted if we act with $SL_2(\mathbb{Z}) = \{ (a \ b) \mid ad - bc \neq 0 \}$ by $z \mapsto \frac{az+b}{cz+d}$

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which is partition function
Monstrous Moonshine

- Special modular function of weight 0 called \( j \)-function:

\[
j(z) = q^{-1} + 744 + 196884q + 21493760q^2 + \ldots
\]

gives change of coordinates on sphere
Monstrous Moonshine

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... gives change of coordinates on sphere

- Character table of Monster group $\mathbb{M}$

<table>
<thead>
<tr>
<th>$\mathbb{M}$</th>
<th>1A</th>
<th>2A</th>
<th>2B</th>
<th>3A</th>
<th>3B</th>
<th>3C</th>
<th>4A</th>
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<td>$M_1$</td>
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<tr>
<td>$M_3$</td>
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<td>91884</td>
<td>-2324</td>
<td>7889</td>
<td>-130</td>
<td>248</td>
<td>1772</td>
<td>...</td>
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<tr>
<td>$M_4$</td>
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<td>1139374</td>
<td>12974</td>
<td>55912</td>
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- 1978: McKay noticed this and sent it to Thompson and he was astounded, called it Moonshine
Explanation

- McKay and Thompson hypothesized a function for each conjugacy class $T_g(z)$
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1988: Harvey et. al in “Beauty and the Beast” showed that a 24-dimensional bosonic torus model has partition function $j(z) - 720$ and symmetry group $\mathbb{M}$, where each $T_g$ corresponds to different boundary conditions.
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Prize question: Does there exist Monster space (manifold)? Professor Harvey and I are looking at candidate $M_{80}$ from physics perspective.

Now there are many types of Moonshine: Umbral, Thompson, Conway, Rudvalis, Baby Monster, Mathieu,... all different!
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K3 Surfaces & Topological Invariants

- String theory: $26D \to 11D \to 10D$, but we only see 4-dimensions so we have $\mathbb{R}^{3,1} \times M$, where $M$ is Calabi-Yau manifold
K3 Surfaces & Topological Invariants

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- Two types of 4-dimensional Calabi-Yau, one is K3 surface
  - Consider a 4-dimensional torus $\mathbb{T}^4$ and identify $x \sim -x$ to get $M = \mathbb{T}^4/\mathbb{Z}_2$

- 2010: Eguchi, Ooguri, Tachikawa notice $A(n)$ is dimension of $n$th irreducible representation of $M_{24}$

- Gannon showed true for all $n$ and for $A_g(n)$, where we twist states by action of $g \in M_{24}$, but it has been shown no K3 theory has $M_{24}$ symmetry!
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▶ Decompose into massless and massive parts
  $$Z_{K3}(\tau, z) = 20 \text{ch}_{h = \frac{1}{4}, l = 0}(\tau, z) - 2 \text{ch}_{h = \frac{1}{4}, l = \frac{1}{2}}(\tau, z) + \sum_{n=1}^{\infty} A(n) \text{ch}_{h = n + \frac{1}{4}, l = \frac{1}{2}}(\tau, z)$$
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Discriminant Property

Expand massive part of $Z_{K3}$

$$\sum_{n=1}^{\infty} A(n) \operatorname{ch}_{h=n+\frac{1}{4}, l=\frac{1}{2}}(\tau, z) = 2 \frac{\theta_1(\tau, z)}{\eta^3(\tau)} (-q^{-1/8} + 45q^{7/8} + 231q^{15/8} + \ldots)$$

Conjecture: The corresponding representation given whose dimension is coefficient of $q^D/8$ contains at least one pair of such $\rho_n$, $\rho^* n$.

Physically, this says that the energy of states tells us in which representations they transform.

Checked to very high order and for 6 different types of Moonshine—want to find some underlying physical explanation of why this is true.
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- Look at exponent $D/8$; if $D = n\lambda^2$ with $n$ and $\lambda$ coprime, then there exists representation $\rho$ and $g \in M_{24}$ such that
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